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Metric Projections after Renorming

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A multi-valued mapping of a reflexive real Banach space into its subspace is a metric projection for a suitable equivalent norm iff it has non-empty closed convex values, is norm-to-weak upper semi-continuous, and is semi-linear. As an application of this characterization we prove that, given an infinite-dimensional subspace of codimension at least two in a reflexive space, there exists an equivalent norm such that the subspace is Chebyshev but the metric projection is not continuous.
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INTRODUCTION

Let M be a closed subspace of a real normed linear space X. A multi-valued mapping $P: X \to 2^M$ is called *a metric projection after renorming* if there exists an equivalent norm $|||\cdot|||$ on X such that P is equal to the metric projection onto M in $(X, |||\cdot|||)$.

In his interesting paper [2], A. L. Brown gave a characterization of metric projections after renorming in the case of finite-dimensional X. The present paper is devoted to the investigation of metric projections after renorming in infinite-dimensional spaces.

In Section 1 we give a characterization of parts (i.e., multi-valued selections) of metric projections after renorming. The main result of this paper is contained in Section 2. Theorem 2.6 asserts that, if X is reflexive, $P: X \to 2^M$ is a metric projection after renorming if and only if it has non-empty closed convex values, it is norm-to-weak upper semi-continuous, and semi-linear with respect to M (see Definition 1.1(i)). As an application of this result we prove in Section 3 that, given an infinite-dimensional subspace of codimension greater than one, a reflexive Banach space can be equivalently renormed so that the subspace be Chebyshev and the metric projection be not continuous (Theorem 3.4). Till now, there have been known only several examples of subspaces in reflexive spaces with discon-

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tinuous metric projections [1, 7, 5, 8], all of them dealing with a suitable renorming of l_2. Our result states that such an example provides any subspace (with trivial exceptions) of any infinite-dimensional reflexive space (after a suitable renorming).

Let us state some notations and definitions. For a multi-valued mapping $P: X \to 2^M$ we shall denote $D(P) = \{x \in X;\ P(x) \neq \emptyset\}$ (the *domain* of P) and we shall often write $P(x) = y$ instead of $P(x) = \{y\}$. We shall say that P is a part of $Q: X \to 2^M$ if $P(x) \subset Q(x)$ for any $x \in X$.

P is called *norm-to-weak upper semi-continuous* ($(n-w)$ *usc*) if for any $x \in X$ and any weakly open set V with $P(x) \subset V$ there exists an open (in the norm topology) neighborhood U of x with $P(U) \subset V$.

We shall denote by X/M the quotient space, by Q_M the quotient mapping $x \mapsto x + M$, and $M^- = \{f \in X^*;\ f(m) = 0$ for any $m \in M\}$. The *metric projection* of X onto M is the multi-valued mapping which sends each point $x \in X$ to the set of best approximations (nearest points) to x in M. The subspace M is called *proximinal* if each point of X lies in the domain of the metric projection onto M. If in addition the metric projection is single-valued then M is called *Chebyshev*.

The *duality mapping* on X is the mapping $J: X \to 2^{X^*}$ defined by $J(x) = \{f \in X^*,\ f(x) = \|f\| \cdot \|x\|$ and $\|f\| = \|x\|\}$. By the Hahn–Banach Theorem $D(J) = X$. It is a well-known fact that J is (n-w)usc in reflexive spaces. The norm on X is *smooth* (*Fréchet smooth*, resp.) if the duality mapping J is single-valued (single-valued and continuous, resp.).

A closed affine subset A of X is *tangent* to a convex set K at a point x if $x \in K \cap A \subset \partial K$, where ∂K is the boundary of K.

All (normed) linear spaces in this paper are real.

1. Parts of Metric Projections after Renorming

DEFINITION 1.1. Let M be a subspace of a linear space X, $P: X \to 2^M$.

(i) P is called *semi-linear with respect to M* (*w.r.t. M*) if

$$P(kx + m) = kP(x) + m, \qquad \text{whenever } k \in \mathbb{R},\ x \in X,\ m \in M.$$

(ii) The mapping P determines another mapping $\hat{P}: X \to 2^M$, defined by

$$\hat{P}(x) = \bigcup_{t \neq 0,\, u \in M} \frac{1}{t}\,(P'(tx + u) - u) = \bigcup_{t \neq 0,\, u \in M} \frac{1}{t} \cdot P'(tx + tu) - u,$$

where $P'(x) = P(x)$ for $x \in X \setminus M$, $P'(x) = P(x) \cup \{x\}$ for $x \in M$.

Remark 1.2. (a) If P is semi-linear w.r.t. M, then $P(m)=m$ for any $m\in M$ by Definition 1.1(i).

(b) It is clear that $P\subset\hat{P}$, and $\bigcup_{t\in\mathbb{R},\, x\in D(P)}(tx+M)=D(\hat{P})$ if $D(P)\neq\emptyset$.

LEMMA 1.3. *Let M, X, P be as in Definition* 1.1. *Then the following are equivalent.*

(i) *P is a part of a semi-linear (w.r.t. M) mapping $S\colon X\to 2^M$.*

(ii) *$\hat{P}$ is semi-linear w.r.t. M.*

(iii) *$P(m)\subset\{m\}$ for any $m\in M$.*

Moreover, if the conditions above hold, $\hat{P}$ is the minimal semi-linear extension of P.

Proof. (a) The implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) follow immediately from Remark 1.2.

(b) Let (iii) hold. Then $P'(m)=m$, and therefore also $\hat{P}(m)=m$, for any $m\in M$. Let $k\in\mathbb{R}\setminus\{0\}$, $x\in X$, $m\in M$. Then

$$\begin{aligned}\hat{P}(kx+m)&=\bigcup_{t\neq 0,\, u\in M}\frac{1}{t}\,(P'(t(kx+m)+u)-u)\\&=\bigcup_{t\neq 0,\, u\in M}k\cdot\frac{1}{kt}\,(P'(ktx+(tm+u))-(tm+u))+m\\&=k\hat{P}(x)+m\end{aligned}$$

and the implication (iii) $\Rightarrow$ (ii) is proved.

(c) Let P be a part of a semi-linear mapping S. By (iii) and Remark 1.2(b), also P' is a part of S. Hence

$$\begin{aligned}\hat{P}(x)&=\bigcup_{t\neq 0,\, u\in M}\frac{1}{t}\,(P'(tx+u)-u)\\&\subset\bigcup_{t\neq 0,\, u\in M}\frac{1}{t}\,(S(tx+u)-u)=S(x)\end{aligned}$$

by the semi-linearity of S. ∎

Now we are prepared to characterize parts of metric projections after renorming.

THEOREM 1.4. *Let M be a closed linear subspace of a normed linear space X and let $P: X \to 2^M$. Then the following assertions are equivalent.*

(i) *P is a part of a metric projection after renorming.*

(ii) *P is a part of a semi-linear (w.r.t M) mapping which is locally bounded at the origin, and $D(P)$ is in the domain of a metric projection after renorming.*

(iii) *$P(m) \subset \{m\}$ for any $m \in M$,*

$$\inf_{\delta > 0} \sup\{\|z-u\| \cdot |t|^{-1};\ t \in \mathbb{R} \setminus \{0\},\ u \in M,\ \|x\| < \delta,\ z \in P(tx+u)\} < \infty, \tag{1}$$

and $D(P)$ is in the domain of a metric projection after renorming.

Proof. (a) Each metric projection onto M is semi-linear w.r.t. M and locally bounded at the origin (see, e.g., [6]). This proves (i) $\Rightarrow$ (ii).

(b) The equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 1.3 and from the easy fact that the condition (1) is equivalent to the local boundedness of $\hat{P}$ at the origin.

(c) Let P satisfy (ii) and let $|\cdot\|$ be an equivalent norm on X such that $D(P) \subset D(\pi)$, where $\pi: X \to 2^M$ is the metric projection in $(X, \|\cdot\|)$. The semi-linearity of π, together with Remark 1.2(b), implies $D(\hat{P}) \subset D(\pi)$. Denote by B the unit ball in $(X, \|\cdot|)$ and put $\Sigma = \partial B \cap \pi^{-1}(0)$, $K_0 = (1/2)\, B \cup (I-\hat{P})(\Sigma)$, $K = \overline{\mathrm{co}}\, K_0$. The set K is closed and convex, and $0 \in \operatorname{int} K$. $\hat{P}$ is locally bounded at 0 because of its minimality among all semi-linear extensions of P (Lemma 1.3). Hence, by the homogeneity of $\hat{P}$, it is bounded on bounded sets. Therefore K is also bounded and symmetric. This shows that K is the unit ball for an equivalent norm $|||\cdot\|$ on X.

Let $x \in \Sigma$ be arbitrary. Then $\|x-m\| \geqslant \|x-\pi(x)\| = \|x\| = 1$ for any $m \in M$. Thus the affine space $x + M$ is tangent to B at x. By the Hahn–Banach Theorem, there exists $f \in J(x) \cap M^{\perp}$, where J is the duality mapping in $(X, \|\cdot\|)$. Now

$$f(x-\hat{P}(x)) = f(x) = 1, \qquad f(\xi - \hat{P}(\xi)) = f(\xi) \leqslant 1 \qquad \text{for any} \quad \xi \in \Sigma.$$

$$f(y) \leqslant \tfrac{1}{2} \qquad \text{for any} \quad y \in \tfrac{1}{2}B.$$

These facts imply that $f^{-1}(1)$, and hence also $x + M$, is tangent to K at the points of $x - \hat{P}(x)$. In other words: $\hat{P}(x) \subset \tilde{\pi}(x)$ for $x \in \Sigma$, where $\tilde{\pi}: X \to 2^M$ is the metric projection in $(X, \|\cdot\|)$.

Now let $x \in X$ be arbitrary. If $x \in (X \setminus D(\hat{P})) \cup M$ then clearly

$\hat{P}(x) \subset \tilde{\pi}(x)$. If $x \in D(\hat{P}) \setminus M$, then $(x - \pi(x))\, \|x - \pi(x)\|^{-1}$ is a subset of Σ, and hence

$$\begin{aligned}\hat{P}(x) &= \hat{P}\left(\|x - \pi(x)\| \cdot \frac{x - \pi(x)}{\|x - \pi(x)\|} + \pi(x)\right)\\ &= \|x - \pi(x)\|\, \hat{P}\left(\frac{x - \pi(x)}{\|x - \pi(x)\|}\right) + \pi(x)\\ &\subset \|x - \pi(x)\|\, \tilde{\pi}\left(\frac{x - \pi(x)}{\|x - \pi(x)\|}\right) + \pi(x) = \tilde{\pi}(x), \qquad (2)\end{aligned}$$

because $\hat{P}$ and $\tilde{\pi}$ are semi-linear w.r.t. M. We have proved that $P(x) \subset \hat{P}(x) \subset \tilde{\pi}(x)$ for any $x \in X$, where $\tilde{\pi}$ is a metric projection after renorming. ▌

Let us note that the idea of how to defines the needed equivalent norm is due to A. L. Brown [2], who used it in $\mathbb{R}^n$.

2. Metric Projections after Renorming

Definition 2.1. Let A be a subset of a normed linear space X. We shall say that an $f \in X^*$ *strongly exposes* A *at a point* $x \in A$, if $f(x) = \sup f(A)$, and $x_n \to x$ whenever $\lim f(x_n) = f(x)$ and $x_n \in A$.

First we shall state an equivalence of two geometric conditions. Since it is not substantial for further results, we omit the straightforward proof, which uses only the identification of $(X/M)^*$ with $M^{\perp}$ and the following theorem of Holmes.

Theorem 2.2 [6]. *Let $\pi: X \to 2^M$ be the metric projection of a normed linear space X onto a Chebyshev subspace M. Then π is continuous iff the restriction to $\Sigma = \partial B \cap \pi^{-1}(0)$ of the quotient mapping Q_M is a homeomorphism onto the unit sphere in X/M. (B denotes the unit ball in X.)*

Lemma 2.3. *Let M be a proximinal subspace of a normed linear space X. Let $\pi: X \to 2^M$ be the metric projection onto M and let $\Sigma = \partial B \cap \pi^{-1}(0)$, where B is the unit ball in X. Then the following statements are equivalent.*

(i) *For any $x \in \Sigma$, there exists an $f \in J(x) \cap M^{\perp}$ which strongly exposes Σ at x.*

(ii) *M is Chebyshev with π continuous and the unit sphere in X/M is strongly exposed at each of its points.*

THEOREM 2.4. *Let M be a (closed) proximinal subspace of a normed linear space X and let π be the metric projection onto M. Let the condition* (i) (or (ii), *equivalently*) *of Lemma* 2.3 *be satisfied. Let $P: X \to 2^M$ satisfy the properties*

(i) *$P(x)$ is non-empty, closed, and convex for any $x \in X$,*

(ii) *P is (n-w) usc,*

(iii) *P is semi-linear w.r.t. M.*

Then P is a metric projection after renorming.

Proof. (a) First observe that the homogeneity of P and its norm-to-weak upper semi-continuity at 0 imply its local boundedness at 0. Therefore there exists a positive constant L such that

$$\sup \|P(x)\| \leqslant L \|x\| \qquad \text{for any} \quad x \in X, \tag{3}$$

again by the homogeneity of P.

Using the notation of Lemma 2.3, define $K_0 = (1/2)B \cup (I-P)(\Sigma)$, $K = \overline{\text{co}}\, K_0$. As in the proof of Theorem 1.4, K is the unit ball for some equivalent norm $\|\cdot\|$ on X and P is a part of $\tilde{\pi}$, where $\tilde{\pi}$ is the metric projection onto M in $(X, |\!|\!|\cdot|\!|\!|)$.

In order to prove the opposite inclusion $\tilde{\pi} \subset P$, it is sufficient to prove

$$(x+M) \cap K \subset x - P(x) \qquad \text{for any} \quad x \in \Sigma. \tag{4}$$

In fact, (4) implies

$$\begin{aligned} x - \tilde{\pi}(x) &= \|x-\pi(x)\| \cdot \left(\frac{x-\pi(x)}{\|x-\pi(x)\|} - \tilde{\pi}\left(\frac{x-\pi(x)}{\|x-\pi(x)\|} \right) \right) \\ &\subset \|x-\pi(x)\| \left(\left(\frac{x-\pi(x)}{\|x-\pi(x)\|} + M \right) \cap K \right) \\ &\subset \|x-\pi(x)\| \left(\frac{x-\pi(x)}{\|x-\pi(x)\|} - P\left(\frac{x-\pi(x)}{\|x-\pi(x)\|} \right) \right) = x - P(x) \end{aligned}$$

whenever $x \in X \backslash M$. (We have used the fact that the distance of a point of Σ from M in $(X, |\!|\!|\cdot|\!|\!|)$ is equal to 1.)

(b) Let $x \in \Sigma$ be arbitrary and let $f \in J(x) \cap M^{\perp}$ expose Σ strongly at x. Suppose that $z \in (x+M) \cap K$ is such that $z \notin x - P(x)$. The Hahn–Banach Theorem gives the existence of a $g \in X^*$ such that $\|g\| = 1$ and $g(z) > s := \sup g(x - P(x))$. Put $\varepsilon = (g(z) - s)/3$. The mapping $I - P$ is (n-w) usc, hence there exists a $\Delta > 0$ such that $\xi - P(\xi) \subset g^{-1}((-\infty, s+\varepsilon))$

whenever $\|\xi - x\| < \Delta$. This, together with the strong exposing of Σ by f at x, ensures the existence of a $\delta > 0$ with the property

$$\xi - P(\xi) \subset g^{-1}((-\infty, s+\varepsilon)), \qquad \text{whenever } \xi \in \Sigma \text{ and } 1 - f(\xi) < \delta. \quad (5)$$

There exists a sequence $\{z^n\} \subset \operatorname{co} K_0$ converging to z. It is possible to write

$$z^n = \lambda_0^n(\tfrac{1}{2} \cdot b^n) + \sum_{i=1}^{k(n)} \lambda_i^n \cdot (\xi_i^n - p_i^n),$$

where

$$b^n \in B,\ \xi_i^n \in \Sigma,\ p_i^n \in P(\xi_i^n),\ \lambda_i^n \geqslant 0,\ \sum_{i=0}^{k(n)} \lambda_i^n = 1.$$

Now $\lambda_0^n \to 0$, because

$$\begin{aligned} 1 = f(x) = f(z) &= \lim f(z^n) \\ &= \lim \left((\lambda_0^n/2)\, f(b^n) + \sum_{i=1}^{k(n)} \lambda_i^n f(\xi_i^n) \right) \\ &\leqslant \liminf \left(\lambda_0^n/2 + \sum_{i=1}^{k(n)} \lambda_i^n \right) \\ &= \liminf (1 - \lambda_0^n/2) \leqslant \limsup (1 - \lambda_0^n/2) \leqslant 1. \end{aligned}$$

Putting $\delta_n = (1 - f(z^n))^{1/2}$, we have $\delta_n \to 0$ and hence $\delta_n \leqslant \delta$ for $n \geqslant n_0$. Let us denote

$$I_n = \{1, 2, ..., k(n)\}, \qquad A_n = \{i \in I_n;\ 1 - f(\xi_i^n) \geqslant \delta_n\}. \quad (6)$$

For any $n \in \mathbb{N}$,

$$\begin{aligned} (\delta_n)^2 = 1 - f(z^n) &= \lambda_0^n(1 - f(b^n)/2) + \sum_{i \in I_n} \lambda_i^n (1 - f(\xi_i^n)) \\ &\geqslant \sum_{i \in A_n} \lambda_i^n (1 - f(\xi_i^n)) \geqslant \sum_{i \in A_n} \lambda_i^n \delta_n. \end{aligned}$$

Thus $\sum_{i \in A_n} \lambda_i^n \leqslant \delta_n$.

Let $n \geqslant n_0$. Then, by (3), (5), and (6),

$$\begin{aligned} g(z^n) = (\lambda_0^n/2)\, g(b^n) &+ \sum_{i \in A_n} \lambda_i^n g(\xi_i^n - p_i^n) \\ &+ \sum_{i \in I_n \setminus A_n} \lambda_i^n g(\xi_i^n - p_i^n) \end{aligned}$$

$$\leqslant \lambda_0^n/2 + \sum_{i \in A_n} \lambda_i^n(1+L) + \sum_{i \in I_n \setminus A_n} \lambda_i^n(s+\varepsilon)$$

$$= (s+\varepsilon) + \lambda_0^n(1/2 - s - \varepsilon) + \sum_{i \in A_n} \lambda_i^n(1+L-s-\varepsilon)$$

$$\leqslant s + \varepsilon + \lambda_0^n |1/2 - s - \varepsilon| + \delta_n |1 + L - s - \varepsilon|.$$

This implies $g(z^n) \leqslant s + 2\varepsilon = g(z) - \varepsilon$ for sufficiently large n. But this is in contradiction with $z^n \to z$. ▮

Remark 2.5. It is clear from the proof of Theorem 2.4, that the unit ball of the required equivalent norm could be defined by $K = \overline{\text{co}}(\alpha B \cup (I-P)(\Sigma))$ with an arbitrary $\alpha \in (0, 1)$.

As an easy consequence of Theorem 2.4 we get the main result of the present paper.

THEOREM 2.6. *Let M be a closed subspace of a reflexive Banach space X and let $P: X \to 2^M$. Then P is a metric projection after renorming if and only if P has non-empty closed convex values and P is (n-w)usc and semi-linear w.r.t. M.*

Proof. Necessity. Let P be the metric projection onto M in $(X, \|\cdot\|)$. It is easy and well known that P has non-empty closed convex values and P is semi-linear w.r.t. M. Suppose P is not (n-w)usc at a point x. There exist a weakly open set W and sequences $\{x_n\} \subset X$ and $\{y_n\} \subset M$, such that $P(x) \subset W$, $x_n \to x$ and $y_n \in P(x_n) \setminus W$. The sequence $\{y_n\}$ is bounded since $\|y_n - x\| \leqslant \|y_n - x_n\| + \|x_n - x\| = \text{dist}(x_n, M) + \|x_n - x\|$. Using weak compactness, we can suppose that y_n converge weakly to some y, without any loss of generality. Clearly $y \in M$ and $\|x - y\| \leqslant \liminf \|x_n - y_n\| = \liminf \text{dist}(x_n, M) = \text{dist}(x, M)$. Thus $y \in P(x) \subset W$, which is in contradiction with $y_n \notin W$.

Sufficiency. By the Trojanski Renorming Theorem (see [3]), we can suppose that X is equipped with a locally uniformly convex norm. Then the unit ball B of X is strongly exposed at any point $x \in \partial B$ by any $f \in J(x)$. Consequently, the condition (i) of Lemma 2.3 is satisfied with any $f \in J(x) \cap M^\perp$ (such an f exists by the Hahn–Banach Theorem). Clearly M is proximinal since X is reflexive. By Theorem 2.4, P is a metric projection after renorming. ▮

3. DISCONTINUOUS METRIC PROJECTIONS

In this section, the characterization of metric projections after renorming (Theorem 2.6) is applied to the existence of discontinuous single-valued metric projections onto subspaces of a reflexive space after a suitable renorming.

We begin with some lemmas. The first of them is a consequence of C. Franchetti's "(H)-property destroying method."

LEMMA 3.1 [4]. *Every reflexive infinite-dimensional Banach space has an equivalent smooth norm which is not Fréchet smooth.*

LEMMA 3.2. *Every Banach space has an equivalent norm, such that its unit ball is strongly exposed at some point by some element of the dual.*

Proof. Let B be the unit ball of a Banach space X. Take any $x \in \partial B$ and $f \in J(x)$. Then the set $K = \overline{\mathrm{co}}(B \cup \{-2x, 2x\})$ is the unit ball of some equivalent norm on X. It is elementary to see that K is strongly exposed at $2x$ by f. ▮

LEMMA 3.3. *Let X be a Banach space with* $\dim X \geqslant 2$ *and let Y be a reflexive infinite-dimensional Banach space. Then there exists a mapping $F: X \to Y$ which is homogeneous and norm-to-weak continuous, but F is not continuous in norm topologies.*

Proof. (a) By Lemma 3.1, there exists an equivalent norm on Y^*, which is smooth but not Fréchet smooth. Then the duality mapping J^* on Y^* is single-valued, has its values in Y, is norm-to-weak continuous, but is not continuous. Note that J^* is homogeneous. There exists a sequence $\{g_n\} \subset Y^*$ such that $\|g_n\| = 1$, $g_n \to g_0 \in Y^*$ and $J^*(g_n)$ do not converge to $J^*(g_0)$ in norm.

(b) Consider an equivalent norm $\|\cdot\|$ on X such that its unit ball B is strongly exposed at some point $x_0 \in \partial B$ by some $f \in X^*$, $\|f\| = 1$ (Lemma 3.2). Define a continuous mapping $\varphi: [-1, 1] \to Y$ by the properties

$$\varphi(0) = 0,\ \varphi(1) = g_0,\ \varphi\left(\frac{n}{n+1}\right) = g_n \qquad \text{for} \quad n \in \mathbb{N},$$

$$\varphi \text{ is affine on } [0, 1/2] \text{ and on each } \left[\frac{n}{n+1}, \frac{n+1}{n+2}\right],\ n \in \mathbb{N},$$

$$\varphi(t) = -\varphi(-t) \qquad \text{for} \quad t \in [-1, 0].$$

Then the mapping $h: \partial B \to Y^*$, $h(x) = \varphi(f(x))$, is bounded, continuous, and odd. Hence the mapping $H: X \to Y^*$, defined by

$$H(0) = 0, \qquad H(x) = \|x\|\, h\left(\frac{x}{\|x\|}\right) \qquad \text{for} \quad x \neq 0,$$

is continuous and homogeneous.

(c) Put $F=J^* \circ H$. Then F is a homogeneous and norm-to-weak continuous mapping of X into Y. It remains to show that F is not continuous.

For any $n \in \mathbb{N}$ choose an $x_n \in \partial B$ with $f(x_n)=n/(n+1)$ (its existence is assured by the connectedness of ∂B). Since f strongly exposes B at x_0, we have $x_n \to x_0$.

$$F(x_n)=J^*H(x_n)=J^*\left(\varphi\left(\frac{n}{n+1}\right)\right)=J^*(g_n),$$

$$F(x_0)=J^*H(x_0)=J^*(\varphi(1))=J^*(g_0).$$

It follows that $F(x_n)$ do not converge to $F(x_0)$, and hence F is not continuous at x_0. ∎

THEOREM 3.4. *Let M be a closed subspace of a reflexive Banach space X. If* codim $M>1$ *and M is infinite-dimensional, then there exists an equivalent norm on X such that M is Chebyshev and the metric projection onto M is not continuous.*

Proof. Let $\|\cdot\|$ be an equivalent locally uniformly convex norm on X (see [3]). Then the metric projection π onto M is single-valued and continuous. By Lemma 3.3, there exists a mapping $F\colon X/M \to M$ which is homogeneous, norm-to-weak continuous and not continuous. Define $P=F \circ Q_M+\pi$, where Q_M is the quotient mapping. Clearly, P is homogeneous and norm-to-weak continuous. P is also semi-linear w.r.t. M, because

$$\begin{aligned} P(x+m) &= F(Q_M(x+m))+\pi(x+m)=F(Q_M(x))+\pi(x)+m \\ &= P(x)+m \qquad \text{for} \quad x \in X, m \in M. \end{aligned}$$

By Theorem 2.6, P is a metric projection after renorming. It remains to show that P is not continuous.

Take $\xi_n \in X/M$ such that $\xi_n \to \xi \in X/M$ but $F(\xi_n)$ do not converge to $F(\xi)$. Denote by q the restriction on $\pi^{-1}(0)$ on Q_M. Then, by Holmes' theorem (Theorem 2.2), q is a homeomorphism of $\pi^{-1}(0)$ onto X/M. Then the points $x_n=q^{-1}(\xi_n)$ converge to the point $x=q^{-1}(\xi)$, but $P(x_n)=F(\xi_n)$ do not converge to $P(x)=F(\xi)$. This shows that P is not continuous at x. ∎

COROLLARY 3.5. *Let M be a closed subspace of a reflexive Banach space. Then M is the range of a discontinuous single-valued metric projection after renorming, if and only if M is infinite-dimensional and M is not a hyperplane in X.*

Proof. The assertion is a direct consequence of Theorem 3.4 and of the well-known fact that metric projections onto Chebyshev finite-dimensional subspaces or onto Chebyshev hyperplanes in a reflexive space are continuous. ∎

Considering Brown's example [1], it is natural to ask the following

Problem. Does there exist an equivalent norm from Theorem 3.4 which would be in addition strictly convex?

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